

BERNOULLI NUMBERS AND DEFORMATIONS OF SCHEMES AND MAPS

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ABSTRACT. We introduce a notion of Jacobi-Bernoulli cohomology associated to a semi-simplicial Lie algebra (SELA). For an algebraic scheme X over \mathbb{C} , we construct a tangent SELA \mathcal{T}_X and show that the Jacobi-Bernoulli cohomology of \mathcal{T}_X is related to infinitesimal deformations of X .

0. OVERVIEW

The 'usual' deformation theory, e.g. of complex structures, in the manner of Kodaira-Spencer-Grothendieck (cf. e.g. [3, 9] and references therein), is commonly couched in terms of a differential graded Lie algebra or dgla \mathfrak{g} . It can be viewed, as in [7], as studying the *deformation ring* $R(\mathfrak{g})$, defined in term of the *Jacobi cohomology*, i.e. the cohomology of the Jacobi complex associated to \mathfrak{g} . This setting is somewhat restrictive, e.g. it is not broad enough to accomodate such naturally occurring deformation problems as embedded deformations of a submanifold X in a fixed ambient space Y . In [6] we introduced the notion of *Lie atom* (essentially, Lie pair) and an associated *Jacobi-Bernoulli complex* as an extension of that of dgla and its Jacobi complex, one that is broad enough to handle embedded deformations and a number of other problems besides.

A purpose of this paper is to establish the familiar notion of (dg) *semi-simplicial* Lie algebra (SELA) as an appropriately general and convenient setting for deformation theory. As a first approximation, one can think of SELA as a structure like that of the Čech complex of a sheaf of Lie algebras on a topological space X with respect to some open covering of X . Not only is SELA a broad generalization of Lie atom, it is broad enough, as we show, to encompass deformations of (arbitrarily singular) algebraic schemes (over \mathbb{C}).

To express the deformation theory of a SELA \mathfrak{g}_\bullet we introduce a complex that we call the Jacobi-Bernoulli complex of \mathfrak{g}_\bullet , though a more proper attribution would be to Jacobi-Bernoulli-Baker-Campbell-Hausdorff. In a nutshell, the point of this complex is that it transforms a gluing condition from nonabelian cocycle condition to ordinary (additive) cocycle condition via the multilinearity of the groups making up the complex. A typical gluing

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condition looks like

$$(0.1) \quad \Psi_{\alpha\beta}\Psi_{\beta\gamma}\Psi_{\gamma\alpha} = 1$$

with $\Psi_{\alpha\beta} \in \exp(\mathfrak{g}_{\alpha\beta})$, where $\mathfrak{g}_{\alpha\beta}$ may be thought of as the component of our SELA \mathfrak{g}_\bullet having to do with gluing over $U_\alpha \cap U_\beta$. This condition can be transformed as follows. Write

$$\Psi_{\alpha\beta} = \exp(\psi_{\alpha\beta})$$

etc. Now the BCH formula gives a formal expression

$$\exp(X)\exp(Y)\exp(Z) = \sum W_{i,j,k}(X,Y,Z)$$

where $W_{i,j,k}(X,Y,Z)$ is a homogeneous ad-polynomial of tridegree i,j,k ('BCH polynomial'), which can be viewed as a linear map

$$w_{ijk} : \text{Sym}^i(\mathfrak{g}_{\alpha\beta}) \otimes \text{Sym}^j(\mathfrak{g}_{\beta\gamma}) \otimes \text{Sym}^k(\mathfrak{g}_{\gamma\alpha}) \rightarrow \mathfrak{g}_{\alpha\beta\gamma}$$

Then (0.1) becomes the additive condition that

$$w_{i,j,k}(\psi_{\alpha\beta}^i \otimes \psi_{\beta\gamma}^j \otimes \psi_{\gamma\alpha}^k) = 0, \quad \forall i, j, k.$$

Now our Jacobi-Bernoulli complex $J(\mathfrak{g}_\bullet)$ for the SELA \mathfrak{g}_\bullet is essentially designed so as to encompass the various BCH polynomials w_{ijk} . It is a comultiplicative complex whose groups essentially constitute the symmetric algebra on \mathfrak{g}_\bullet and whose maps are essentially derived from the w_{ijk} by the requirement of comultiplicativity. The dual of the cohomology of $J(\mathfrak{g}_\bullet)$ yields the deformation ring associated to the SELA \mathfrak{g}_\bullet .

As mentioned above, our other main result here is that the deformation theory of an algebraic scheme over \mathbb{C} can be expressed in terms of a SELA. Unsurprisingly, this is done via an affine covering. Thus the first step is to associate a dgla to a closed embedding

$$X \rightarrow P$$

where P is an affine (or for that matter, projective) space. We call this the *tangent* dgla to X and denote it $\mathcal{T}_X(P)$. In a nutshell, $\mathcal{T}_X(P)$ is defined as the mapping cone of a map that we construct

$$T_P \otimes \mathcal{O}_X \rightarrow N_{X/P}$$

where $N_{X/P}$ is the normal atom to X in P as in [6]. That is, $\mathcal{T}_X(P)$ is represented by the mapping cone of a map of free modules representing $T_P \otimes \mathcal{O}_X$ and $N_{X/P}$. We will show $\mathcal{T}_X(P)$ admits a dgla structure, a dgla action on \mathcal{O}_X , as well as \mathcal{O}_X -module structure. Up to a certain type of 'weak equivalence', the dgla $\mathcal{T}_X(P)$ depends only on the isomorphism class of X and not on the embedding in P .

The partial independence on the embedding is good enough to enable us to associate a global SELA $\mathcal{T}_{X\bullet}$ for an arbitrary algebraic scheme X defined in terms of, but up to weak equivalence independent of, an affine covering X_α and embeddings of each X_α in an affine space P_α : e.g.

$$\mathcal{T}_{X,\alpha} = \mathcal{T}_{X_\alpha}(P_\alpha),$$

$$\mathcal{T}_{\mathcal{X},\alpha\beta} = \mathcal{T}_{\mathcal{X}_\alpha \cap \mathcal{X}_\beta}(P_\alpha \times P_\beta)$$

etc. Global deformations of X then amount to a collections of deformations of each X_α , given via Kodaira-Spencer theory by a suitable element $\phi_\alpha \in \mathcal{T}_{X,\alpha}^1$, plus a collection of gluing data $\psi_{\alpha\beta} \in \mathcal{T}_{X,\alpha\beta}^0$, and the necessary compatibilities are readily expressed as a cocycle condition in the Jacobi-Bernoulli complex $J(\mathcal{T}_{X\bullet})$.

1. SEMI-SIMPLICIAL LIE ALGEBRAS AND JACOBI-BERNOULLI COMPLEX

1.1. SELA. Our notion of SELA is essentially the dual of the portion of the usual notion of simplicial Lie algebra involving only the face maps without degeneracy. Let A be a totally ordered index-set. A *simplex* in A is a finite nonempty subset $S \subset A$, while a *biplex* is a pair $(S_1 \subset S_2)$ of simplices with $|S_1| + 1 = |S_2|$; similarly for triplex $(S_1 \subset S_2 \subset S_3)$ etc. The *sign* $\epsilon(S_1, S_2)$ of a biplex (S_1, S_2) is defined by the condition that

$$\epsilon((0, \dots, \hat{p}, \dots, n), (0, \dots, n)) = (-1)^{n-p}.$$

By a *simplicial Lie algebra* (SLA) \mathfrak{g}_\bullet on A we shall mean the assignment for each simplex S on A of a Lie algebra \mathfrak{g}_S , and for each biplex (S_1, S_2) of a map ('coface' or 'restriction')

$$r_{S_1, S_2} : \mathfrak{g}_{S_1} \rightarrow \mathfrak{g}_{S_2}$$

such that $\epsilon(S_1, S_2)r_{S_1, S_2}$ is a Lie homomorphism and such that for each $S_1 \subset S_3$ with $|S_1| + 2 = |S_3|$, we have

$$(1.2) \quad \sum_{\substack{\text{triplex} \\ (S_1 \subset S_2 \subset S_3)}} r_{(S_2, S_3)} r_{(S_1, S_2)} = 0.$$

The identity (1.2) implies that we may assemble the \mathfrak{g}_S into a complex $K(\mathfrak{g}_\bullet)$ where

$$K^i(\mathfrak{g}_\bullet) = \bigoplus_{|S|=i+1} \mathfrak{g}_S$$

and differential constructed from the various $r_{(S_1, S_2)}$.

Example 1.1. If \mathfrak{g} is a sheaf of Lie algebras on a topological space X , and (U_α) is an open covering of X , there is a Čech SELA

$$S \mapsto \mathfrak{g}(\bigcap_{\alpha \in S} U_\alpha).$$

The standard complex $K(\mathfrak{g}_\bullet)$ is in this case the Čech complex $\check{C}(\mathfrak{g}, (U_\alpha))$. This plays a fundamental role in the study of \mathfrak{g} -deformations.

Example 1.2. If $\mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie pair (more generally, a Lie atom, cf. [6]), we get a SELA \mathfrak{g} on (01) with $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = 0$, $\mathfrak{g}_{01} = \mathfrak{h}$.

$$\mathfrak{g} \bullet \text{---} \bullet 0$$

The deformation-theoretic significance of \mathfrak{g} is like that of the Lie atom $(\mathfrak{g}, \mathfrak{h})$, viz. \mathfrak{g} -deformations together with an \mathfrak{h} -trivialization.

An obvious generalization would be to take a pair of maps $\mathfrak{g}_1 \rightarrow \mathfrak{h}, \mathfrak{g}_2 \rightarrow \mathfrak{h}$ (e.g. twice the same map), which corresponds to pairs (\mathfrak{g}_1 - deformation, \mathfrak{g}_2 -deformation) that become equivalent as \mathfrak{h} -deformations.

1.2. Bernoulli numbers and Baker-Campbell-Hausdorff. Let \mathfrak{g} be a nilpotent Lie algebra. For an element $X \in \mathfrak{g}$ we consider the formal exponential $\exp(X)$ as an element of the formal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. Then we can write

$$(1.3) \quad \exp(X) \exp(Y) = \exp(\beta(X, Y))$$

where β is a certain bracket-polynomial in X, Y , known as the Baker-Campbell-Hausdorff or BCH polynomial. We denote by $\beta_{i,j}, \beta_i$ the portion of β in bidegree i, j (resp. total degree i). Note that each $\beta_{i,j}(X, Y)$ will be a linear combination of (noncommutative) ad monomials with a total of i many X 's and j many Y 's. We write such a monomial in the form

$$(1.4) \quad \text{ad}_S(X^i Y^j) = \text{ad}(T_1) \circ \dots \circ \text{ad}(T_{i+j-1})(T_{i+j})$$

where $S \subset [1, i+j]$ is a subset of cardinality i and $T_k = X$ (resp. $T_k = Y$) iff $k \in S$ (resp. $k \notin S$). We denote by

$$(1.5) \quad \text{ad}_S(X_1, \dots, X_i, Y_1, \dots, Y_j)$$

the analogous function, obtained by replacing the x th occurrence of X (resp. y th occurrence of Y) by an X_x (resp. Y_y) and by $\text{ad}_S^{\text{Sym}}(X_1, \dots, X_i, Y_1, \dots, Y_j)$ the corresponding symmetrized version, i.e.

$$(1.6) \quad \text{ad}_S^{\text{Sym}}(X_1, \dots, X_i, Y_1, \dots, Y_j) = \sum_{\substack{\pi \in \mathfrak{S}_i \\ \rho \in \mathfrak{S}_j}} \frac{1}{i!j!} \text{ad}_S(X_{\pi(1)}, \dots, X_{\pi(i)}, Y_{\rho(1)}, \dots, Y_{\rho(j)}).$$

We will compute β , following [8], §2.15 (where Varadarajan attributes the argument to lectures of Bargmann that follow original papers by Baker and Hausdorff). Set

$$D(x) = \frac{e^x - 1}{x}, C(x) = 1/D(x).$$

Thus, $C(x)$ is the generating function for the Bernoulli numbers B_n , i.e.

$$C(x) = 1 + \sum_{n=1}^{\infty} \frac{B_n}{n!} x^n = \sum_{n=0}^{\infty} C_n x^n.$$

Now the reader can easily check that for any derivation ∂ we have

$$\partial \exp(U) \exp(-U) = D(\text{ad}(U))(\partial U), \exp(-U) \partial \exp(U) = D(-\text{ad}(U))(\partial U).$$

Now differentiate (1.3) with respect to X and multiply both sides by $\exp(-\beta(X, Y))$. This yields (where ∂_X is the unique derivation taking X to X and Y to 0)

$$X = \partial_X(\exp(X)) \exp(-X) =$$

$$= \partial_X(\exp(\beta(X, Y))) \exp(-\beta(X, Y)) = D(\text{ad}(\beta(X, Y))) (\partial_X \beta(X, Y)).$$

Thus

$$(1.7) \quad \partial_X \beta(X, Y) = C(\text{ad}(\beta(X, Y)))(X).$$

Similarly,

$$(1.8) \quad \partial_Y \beta(X, Y) = C(-\text{ad}(\beta(X, Y)))(Y).$$

Starting from $\beta_0 = 0$, the formulas (1.7), (1.8) clearly determine β . For example, clearly $\beta_{0,*}(X, Y) = Y$, therefore it follows that

$$(1.9) \quad \beta_{1,*}(X, Y) = C(\text{ad}(Y))(X) = X + \frac{1}{2}[X, Y] + \frac{1}{12}\text{ad}(Y)^2(X) + \dots$$

We shall require the obvious extension of this set-up to the trivariate case. Thus define a function $\beta(X, Y, Z)$ (NB same letter as for the bivariate version) by

$$(1.10) \quad \exp(X) \exp(Y) \exp(Z) = \exp(\beta(X, Y, Z))$$

and let $\beta_{i,j,k}$ denote its portion in tridegree (i, j, k) . Note that

$$\beta(X, Y, Z) = \beta(\beta(X, Y), Z).$$

1.3. Jacobi-Bernoulli complex. Let \mathfrak{g}_\bullet be a SELA. For simplicity, we shall assume \mathfrak{g}_\bullet is 2-dimensional, in the sense that $\mathfrak{g}_S = 0$ for any simplex S of dimension > 2 ; for our applications to deformation theory, this is not a significant restriction. We will also assume that \mathfrak{g}_\bullet is *strongly nilpotent* in the sense that it is an algebra over a commutative ring R such that $\mathfrak{g}_S^{\otimes N} = 0$ for all simplices S and some integer N independent of S , with all tensor products over R . This condition obviously depends only on the S -module structure of \mathfrak{g}_\bullet and not on its Lie bracket. We are going to define a filtered complex $J = J_m^\sharp(\mathfrak{g}_\bullet)$. The groups J^j can be defined succinctly as

$$J^j = (\text{Sym}^\cdot(K^\cdot(\mathfrak{g}_\bullet)[1]))^j$$

where Sym^\cdot is understood in the signed or graded sense, alternating on odd terms, and $K^\cdot(\mathfrak{g}_\bullet)[1]$ is the standard complex on \mathfrak{g}_\bullet shifted left once (which is a complex in degrees $-1, 0, 1$). The increasing filtration F_\cdot is by ‘number of multiplicands’. More concretely,

$$(1.11) \quad J^{j,k} = \bigoplus_{\substack{-\sum_i l_i + \sum_i n_i = j \\ \sum_i l_i + \sum_i m_i + \sum_i n_i = k}} \bigotimes_{i=1}^j \bigwedge_{i=1}^{l_i} \mathfrak{g}_{\alpha_i} \otimes \bigotimes_{i=1}^j \text{Sym}^{m_i} \mathfrak{g}_{\alpha_i \beta_i} \otimes \bigotimes_{i=1}^j \bigwedge_{i=1}^{n_i} \mathfrak{g}_{\alpha_i \beta_i \gamma_i}$$

$$(1.12) \quad F_m J^j = \bigoplus_{k \leq m} J^{j,k},$$

$$(1.13) \quad J^j = F_\infty J^j = F_N J^j.$$

To define the differential d on J^\bullet , we proceed in steps. Let $\alpha < \beta < \gamma$ be indices and recall that we are identifying $\mathfrak{g}_{\gamma\alpha}$ with $\mathfrak{g}_{\alpha\gamma}$.

- The differential is defined so that the obvious inclusion

$$(1.14) \quad K^\bullet(\mathfrak{g}_\bullet)[1]) = F_1 J^\bullet \rightarrow J^\bullet$$

is a map of complexes.

- The component

$$\mathrm{Sym}^i \mathfrak{g}_{\gamma\alpha} \otimes \mathrm{Sym}^j \mathfrak{g}_{\alpha\beta} \otimes \mathrm{Sym}^{\mathfrak{g}}_{\beta\gamma} \rightarrow \mathfrak{g}_{\alpha\beta\gamma}$$

is given by

$$(1.15) \quad X^i Y^j Z^k \mapsto \beta_{i,j,k}(X, Y, Z).$$

- The component

$$\mathfrak{g}_\alpha \otimes \mathrm{Sym}^i \mathfrak{g}_{\alpha\beta} \otimes \mathrm{Sym}^n \mathfrak{g}_{\beta\gamma} \rightarrow \mathrm{Sym}^{i-t+1} \mathfrak{g}_{\alpha\beta} \otimes \mathrm{Sym}^n \mathfrak{g}_{\beta\gamma}, 0 \leq t \leq i$$

is given by

$$(1.16) \quad X \otimes Y^i \otimes Z^n \mapsto C_t Y^{i-t} \mathrm{ad}(Y)^t(X) \otimes Z^n$$

(where C_t is the normalized Bernoulli coefficient).

- Other components are defined subject to the 'derivation rule', e.g. the component

$$\mathfrak{g}_\alpha \otimes \mathrm{Sym}^i \mathfrak{g}_{\beta\gamma} \otimes \mathrm{Sym}^k \mathfrak{g}_{\beta\gamma} \rightarrow \mathfrak{g}_{\gamma\alpha} \otimes \mathrm{Sym}^i \mathfrak{g}_{\alpha\beta} \otimes \mathrm{Sym}^k \mathfrak{g}_{\beta\gamma}$$

is extended in the obvious way from the given differential $\mathfrak{g}_\alpha \rightarrow \mathfrak{g}_{\gamma\alpha}$.

- Components not defined via the above rules are set equal to 0. In particular, the component

$$\mathfrak{g}_\alpha \otimes \mathfrak{g}_{\alpha\beta\gamma} \rightarrow \mathfrak{g}_{\alpha\beta\gamma}$$

is zero.

The following result summarizes the main properties of the Jacobi-Bernoulli complex J associated to a SELA (not least, that it is a complex!). It is in part, but not entirely, a direct extension of the analogous result for Lie atoms given in [6].

Theorem 1.3. (i) (J^\bullet, F_\bullet) is a functor from the category of SELAs over S to that of comultiplicative, cocommutative and coassociative filtered complexes over S .

- (ii) The filtration F_\bullet is compatible with the comultiplication and has associated graded

$$F_i / F_{i-1} = \bigwedge^i(\mathfrak{g}_\bullet).$$

- (iii) The quasi-isomorphism class of $J(\mathfrak{g}_\bullet)$ depends only on the quasi-isomorphism class of \mathfrak{g}_\bullet as SELA.

Proof. As in the proof of [6], Thm 1.2.1, the main issue is to prove J is a complex, i.e. $d^2 = 0$. And again as in [6], it suffices, in light of the derivation rule, to prove the vanishing of the components of d^2 that land in F_1 , i.e. that have just one multiplicative factor. Among those, the proof that these components of d^2 vanish on terms of degree ≤ -2 , i.e. involving $\bigwedge^i \mathfrak{g}_\alpha, i \geq 2$, is similar to the case of the JB complex considered in [6]. The essential new case, not considered in [6], is the vanishing of the F_1 - components of d^2 on terms of degree -1 , i.e. terms of the form

$$X \otimes Y^i \otimes Z^n \in \mathfrak{g}_\alpha \otimes \text{Sym}^i \mathfrak{g}_{\alpha\beta} \otimes \text{Sym}^n \mathfrak{g}_{\beta\gamma}.$$

For such a term, what needs to be shown is the vanishing of the component of d^2 of it in $\mathfrak{g}_{\alpha\beta\gamma}$. Thus, we need to prove that

$$(1.17) \quad d^2(X \otimes Y^i \otimes Z^n)_{\mathfrak{g}_{\alpha\beta\gamma}} = 0.$$

Now this component gets contributions via the various components of $d(X \otimes Y^i \otimes Z^n)$ and those contributions come in two kinds:

- Via $g_{\gamma\alpha} \otimes \text{Sym}^i \mathfrak{g}_{\alpha\beta} \otimes \text{Sym}^n \mathfrak{g}_{\beta\gamma}$, we get $-\beta_{1,i,n}(X, Y, Z)$. This comes from $\beta(\beta(X, Y), Z)$, but is only affected by the terms in $\beta(X, Y)$ of degree ≤ 1 in X , i.e. by

$$U = Y + C(\text{ad}(Y))(X) = Y + \sum_{t=0}^{\infty} C_t \text{ad}(Y)^t(X).$$

This contribution is obtained by taking $-\beta_{i+1-t,n}(U, Z)$ and replacing each monomial

$$\text{ad}_S(U^{i+1-t} Z^n)$$

(cf. (1.4-1.6)) by

$$(i+1-t) \text{ad}_S^{\text{Sym}}(C_t(\text{ad}(Y)^t(X)) Y^{i-t} Z^n)$$

and finally summing as t ranges from 0 to i .

- Via $\text{Sym}^{i+1-t} \mathfrak{g}_{\alpha\beta} \otimes \text{Sym}^n \mathfrak{g}_{\beta\gamma}$, for each $0 \leq t \leq i$, we get a contribution equal to the expression obtained by taking $\beta_{i+1-t,n}(W, Z)$, replacing each monomial $\text{ad}_S(W^{i+1-t} Z^n)$ by $(i+1-t) \text{ad}_S^{\text{Sym}}(C_t \text{ad}(Y)^t(X) Y^{i-t} Z^n)$.

Thus, the sum total of all contributions to $d^2(X \otimes Y^i \otimes Z^n)_{\mathfrak{g}_{\alpha\beta\gamma}}$ is zero. \square

The ring

$$R(\mathfrak{g}_\bullet) = \mathbb{C} \oplus \mathbb{H}^0(J(\mathfrak{g}_\bullet))^*$$

is called the *deformation ring* of \mathfrak{g}_\bullet .

1.4. Special multiplicative cocycles. Let (S, \mathfrak{m}_S) be a local artin \mathbb{C} -algebra and let $\mathfrak{g}_\bullet = \mathfrak{g}_\bullet$ be a dg-SELA (i.e. each \mathfrak{g}_T is a dgla and the coface maps are dg homomorphisms). A special class of (multiplicative) cocycles for the Jacobi-Bernoulli complex

$$J^\bullet(\mathfrak{g}_\bullet \otimes \mathfrak{m}_S) \subset J^\bullet(\mathfrak{g}_\bullet) \otimes \mathfrak{m}_S$$

can be constructed as follows. Suppose

$$\phi_\bullet \in K^0(\mathfrak{g}_\bullet)^1 \otimes \mathfrak{m}_S = \bigoplus_{\rho} \mathfrak{g}_\rho^1 \otimes \mathfrak{m}_S,$$

$$\psi_\bullet \in K^1(\mathfrak{g}_\bullet)^0 \otimes \mathfrak{m}_S = \bigoplus_{\rho\sigma} \mathfrak{g}_{\rho\sigma}^0 \otimes \mathfrak{m}_S$$

are such that, $\forall \rho, \sigma, \tau$,

$$\partial \phi_\rho = -\frac{1}{2}[\phi_\rho, \phi_\rho],$$

$$\partial \psi_{\rho\sigma} = C(\text{ad}(\psi_{\rho\sigma}))(\phi_\rho - \phi_\sigma),$$

$$\beta(\psi_{\rho\sigma}, \psi_{\sigma\tau}, \psi_{\tau\rho}) = 0, \quad .$$

Then let

$$\epsilon(\phi_\bullet, \psi_\bullet) \in J^0(\mathfrak{g}_\bullet) \otimes \mathfrak{m}_S$$

be the element with components

$$\psi_{\rho\sigma} \in \mathfrak{g}_{\rho\sigma} \otimes \mathfrak{m}_S,$$

$$\phi_\rho \in \mathfrak{g}_\rho \otimes \mathfrak{m}_S,$$

and generally

$$\bigwedge^r \phi_\rho \otimes (\psi_{\rho\sigma})^n \in \bigwedge^r (\mathfrak{g}_\rho^1 \otimes \mathfrak{m}_S) \otimes \text{Sym}^n(\mathfrak{g}_{\rho\sigma} \otimes \mathfrak{m}_S), r, n \geq 0.$$

We call $\epsilon(\phi_\bullet, \psi_\bullet)$ a *special multiplicative cocycle with coefficients in S* .

Lemma 1.4. (i) *The cochain $\epsilon(\phi_\bullet, \psi_\bullet)$ defined above is a 0-cocycle for $J(\mathfrak{g}_\bullet)$ and the associated map*

$${}^t\epsilon(\phi_\bullet, \psi_\bullet) \in \text{Hom}(R(\mathfrak{g}_\bullet), S)$$

is a local ring homomorphism.

(ii) *Given S , there is a bijection between cohomology classes of special multiplicative cocycles with coefficients in S and local ring homomorphisms $R(\mathfrak{g}_\bullet) \rightarrow S$.*

□

2. TANGENT SELA

2.1. Affine schemes: tangent dgla. We work in the 'global affine' setting, though the construction can evidently be coherently sheafified. Let X be a closed subscheme of an affine space P and let $I = I_{X/P}$ denote the ideal of X in the coordinate ring A_P (a similar construction can be done for P an arbitrary open subscheme of a projective space). Let

$$(2.18) \quad \dots \rightarrow F^{-1} \rightarrow F^0 \rightarrow I$$

be a free resolution of I . Thus each F^i is a free module on generators e_α^i which correspond for $i = 0$ to generators f_α of I and for $i < 0$ to syzygies $\sum f_{\alpha\beta} e_\beta^{i+1}$.

Next, set $F^1 = A_P$ (the coordinate ring) and let F_+ be the complex in degrees ≤ 1 given by

$$\dots \rightarrow F^0 \xrightarrow{\epsilon} F^1$$

which is a free resolution of A_X . Then set

$$(2.19) \quad \mathcal{N} = \mathcal{N}_{X/P} = \mathcal{H}om(F_+, F_+)$$

which we view as sub-dgla of $\mathcal{E}nd(F_+)$ consisting of maps vanishing on F_+^1 . This may be called the *normal dgla* of X in P . Next, we define a map

$$\kappa : T_P \rightarrow \mathcal{N}^1 = \bigoplus_{i \leq 0} \mathcal{H}om(F_+^i \rightarrow F_+^{i+1})$$

as follows:

- For $v \in T_P$, $\kappa(v)^0$ is the map taking a distinguished generator e_α^0 of F^0 , which corresponds to a generator f_α of I , to $v(f_\alpha)$.
- For $i < 0$, $\kappa(v)^i$ takes a distinguished generator e_α^i of F^i , which corresponds to a syzygy $\sum f_{\alpha\beta} e_\beta^{i+1}$, to $\sum v(f_{\alpha\beta}) e_\beta^{i+1}$.

It is not hard to see that this yields a map of complexes

$$T_P \rightarrow \mathcal{N}[1],$$

i.e. that $d_{\mathcal{N}} \circ \kappa = 0$. Then proceeding similarly, one can lift κ to a map of complexes

$$(2.20) \quad \kappa : T_P \otimes F_+ \rightarrow \mathcal{N}[1]$$

The mapping cone carries a natural structure of dgla which we call the *tangent dgla* of X with reference to P and denote by $\mathcal{T}_X(P)$. It is not hard to see that $\mathcal{T}_X(P)$ is acyclic in negative degrees and bounded. It admits an ' \mathcal{O}_X -module structure' in the form of a pairing

$$F_+ \times \mathcal{T}_X(P) \rightarrow \mathcal{T}_X(P),$$

as well as a 'derivation action' on \mathcal{O}_X (i.e. on F_+).

As for the dependence on the embedding, let $X \rightarrow Q$ be another affine embedding. Then via the diagonal we get a third one $X \rightarrow P \times Q$, together with dgla maps

$$\mathcal{T}_X(P) \rightarrow \mathcal{T}_X(P \times Q), \mathcal{T}_X(Q) \rightarrow \mathcal{T}_X(P \times Q)$$

which, it is easy to check, induce isomorphisms on cohomology in degree ≤ 1 and an injection on H^2 . We call a dgla morphism with these properties a *direct weak equivalence* and define a general *weak equivalence* of dgla's to be a composition of weak equivalences and their inverses.

Example 2.1. If X is a hypersurface with equation f in $P = \mathbb{A}^n$, its tangent dgla may be identified with the complex in degrees $-1, 0, 1$

$$nA_P \xrightarrow{(f,0)} nA_P \oplus A_P \xrightarrow{(\partial f/\partial x_1, \dots, \partial f/\partial x_n, f)} A_P$$

Its H^1 is the so-called Milnor algebra of f (finite-dimensional if f has isolated singularities).

2.2. Maps of affine schemes: tangent dgla. The notion of tangent dgla of an affine scheme can be extended to the case of a *mapping* of affine schemes, as follows. Let

$$f : X \rightarrow Y$$

be a mapping of affine schemes. Given affine embeddings $X \rightarrow P, Y \rightarrow Q$, f can be extended to a map $P \rightarrow Q$. Replacing $X \rightarrow P$ by the graph embedding $X \rightarrow P \times Q$, we may assume $P \rightarrow Q$ is a product projection. Then we have an injection $I_{Y,Q} \rightarrow I_{X,P}$ which extends to the free resolutions $F_{\dot{Y}} \rightarrow F_{\dot{X}}$, and we may moreover assume that each $F_Y^i \otimes A_P \rightarrow F_X^i$ is a direct summand inclusion. We can identify the functor $f^!$ on complexes with $f^! \cdot = \cdot \otimes_{A_Y} F_{X+}$. Then the complex $f^!(\mathcal{N}_{Y/Q})$ can be represented by

$$\mathcal{H}om_{A_Q}(F_{\dot{Y}}, F_{+X}) = \mathcal{H}om_{A_P}(F_{\dot{Y}} \otimes A_P, F_{+X})$$

and there are maps

$$(2.21) \quad \mathcal{N}_{X/P} \rightarrow f^!(\mathcal{N}_{Y/Q}) \leftarrow \mathcal{N}_{Y/Q}$$

The mapping cone of 2.21 can be represented by the sub-dgla of $\mathcal{N}_{X/P} \oplus \mathcal{N}_{Y/Q}$ consisting of pairs (a, b) such that a vanishes on the subcomplex $F_{\dot{Y}} \otimes A_P \subset F_{\dot{X}}$. We denote this mapping cone by \mathcal{N}_f or more properly $\mathcal{N}_{f,P,Q}$ and refer to it as the *normal dgla* of f .

Next, proceeding as in the case of schemes, we can construct a suitable representative of the mapping cone K of

$$T_P \otimes F_{+X} \rightarrow T_Q \otimes F_{+X} \leftarrow T_Q \otimes F_{+Y},$$

together with a map of K to \mathcal{N}_f , so that the mapping cone of $K \rightarrow \mathcal{N}_f$ is a dgla, called the *tangent dgla* to f and denoted \mathcal{T}_f or more properly, $\mathcal{T}_f(P, Q)$. By construction, $\mathcal{T}_f(P, Q)$ is the mapping cone of

$$(2.22) \quad \mathcal{T}_X(P) \oplus \mathcal{T}_Y(Q) \rightarrow f^! \mathcal{T}_Y(Q).$$

2.3. Schemes and maps: tangent SELA. Here we construct the tangent SELA of a separated algebraic scheme X over \mathbb{C} (the separatedness does not seem to be essential). Let (X_ρ) be an affine open covering of X indexed by a well-ordered set, and for each ρ let P_ρ be an affine space with a closed embedding

$$(2.23) \quad \iota_\rho : X_\rho \subset P_\rho.$$

Set $B_\rho = A_{P_\rho}$. We call the system $(X_\rho \subset P_\rho)$ an *affine embedding system* for X . Via the diagonal, $X_\rho \cap X_\sigma$ is a closed subscheme of $X_\rho \times X_\sigma$, hence of $P_\rho \times P_\sigma$. Similarly, for any multi-index $\rho_0 < \dots < \rho_k$, we define

$$(2.24) \quad X_{(\rho_0, \dots, \rho_k)} = X_{\rho_0} \cap \dots \cap X_{\rho_k}, \quad P_{(\rho_0, \dots, \rho_k)} = P_{\rho_0} \times \dots \times P_{\rho_k}$$

and the natural closed embedding

$$(2.25) \quad \iota_{(\rho_0, \dots, \rho_k)} : X_{(\rho_0, \dots, \rho_k)} \subset P_{(\rho_0, \dots, \rho_k)},$$

and we denote the ideal of the latter by $I_{(\rho_0, \dots, \rho_k)}$. We call the system

$$(2.26) \quad (X_{(\rho_0, \dots, \rho_k)} \subset P_{(\rho_0, \dots, \rho_k)}, (\rho_0 < \dots < \rho_k), k \geq 0)$$

the *simplicial extension* of the affine embedding system $(X_\rho \subset P_\rho)$. Note that the defining equations for the image of $\iota_{(\rho_0, \dots, \rho_k)}$ consist of defining equations for the images of individual embeddings ι_{ρ_i} , together with equations for the small diagonal on X^{k+1} . The latter are of course generated by the pullbacks of the equations of the small diagonal in X^k via the various coordinate projections $X^{k+1} \rightarrow X^k$. Therefore, it is possible to choose mutually compatible free resolutions for all the $I_{(\rho_0, \dots, \rho_k)}$, and we denote these by $F_{(\rho_0, \dots, \rho_k)}$. In fact, we may assume that

$$(2.27) \quad F_{(\rho_0, \dots, \rho_k)}^1 = B_{(\rho_0, \dots, \rho_k)} := \bigotimes_{i=0}^k B_{\rho_i},$$

$$(2.28) \quad F_{(\rho_0, \dots, \rho_k)}^i = \bigoplus_{j=0}^k (F_{\rho_j}^i \otimes B_{(\rho_0, \dots, \rho_k)}) \oplus \Delta_{(\rho_0, \dots, \rho_k)}^i, i \leq 0,$$

where $\Delta_{(\rho_0, \dots, \rho_k)}$ is a lifting to $B_{(\rho_0, \dots, \rho_k)}$ of a free resolution of the small diagonal

$$X_{(\rho_0, \dots, \rho_k)} \subset \prod_{j=0}^k X_{\rho_j} \text{ and moreover for any biplex}$$

$$\rho^k = (\rho_0, \dots, \rho_k) \subset \rho^{k+1} = (\rho_0, \dots, \rho_{k+1}),$$

if we let

$$\pi_{\rho^{k+1}, \rho^k} : P_{\rho^{k+1}} \rightarrow P_{\rho^k}$$

denote the natural projection, then we have a direct summand inclusion

$$(2.29) \quad \pi_{\rho^{k+1}, \rho^k}^* F_{\rho^k} := \pi_{\rho^{k+1}, \rho^k}^{-1} F_{\rho^k} \otimes B_{\rho^{k+1}} \rightarrow F_{\rho^{k+1}}$$

Putting together these groups and maps, and twisting by the appropriate sign, i.e. $\epsilon(\rho^k, \rho^{k+1})$, we get an 'extrinsic Čech (double) complex'

$$(2.30) \quad \mathrm{XC}(\mathcal{O}_X) : \bigoplus_{\rho^0} F_{+\rho^0}^\bullet \rightarrow \dots \rightarrow \bigoplus_{\rho^k} F_{+\rho^k}^\bullet \rightarrow \dots$$

Actually this is quasi-isomorphic to the usual Čech complex of \mathcal{O}_X , but we can do more with it. Note that the map 2.29 gives rise to a direct summand inclusion

$$\pi_{\rho^{k+1}, \rho^k}^*(\mathfrak{gl}^\bullet(I_{\rho^k})) \rightarrow \mathfrak{gl}^\bullet(I_{\rho^{k+1}}),$$

whence a dgla map

$$\delta_{\rho^k, \rho^{k+1}} : \mathcal{N}_{X_{\rho^k}/P_{\rho^k}} \rightarrow \mathcal{N}_{X_{\rho^{k+1}}/P_{\rho^{k+1}}}$$

Then we can similarly construct a 'normal SELA'

$$\mathcal{N}_{X_\bullet/P_\bullet} : \dots \rightarrow \bigoplus_{\rho^k} \mathcal{N}_{X_{\rho^k}/P_{\rho^k}} \rightarrow \dots$$

Likewise, we have an 'ambient tangent complex' T_{P_\bullet} and $T_{P_\bullet} \otimes \mathrm{XC}(\mathcal{O}_X)$ and a map

$$(2.31) \quad T_{P_\bullet} \otimes \mathrm{XC}(\mathcal{O}_X) \rightarrow \mathcal{N}_{X_\bullet/P_\bullet}.$$

Finally, we define the *tangent SELA* of X (with reference to the simplicial system (X_\bullet, P_\bullet)) to be the mapping cone of this, and denote it by $\mathcal{T}_{X_\bullet}(X_\bullet, P_\bullet)$ or simply \mathcal{T}_{X_\bullet} . This is the SELA whose value on the simplex ρ^k is the dgla $\mathcal{T}_{X_{\rho^k}}(P_{\rho^k})$. By Theorem 1.3, there is an associated Jacobi-Bernoulli complex $J(\mathcal{T}_{X_\bullet})$, which we denote by J_X and refer to as the *Jacobi-Bernoulli complex of X* . Up to filtered, comultiplicative quasi-isomorphism, it depends only on the isomorphism class of X as scheme over \mathbb{C} . Therefore the *deformation ring* of X

$$R_X = \mathbb{C} \oplus \mathbb{H}^0(J_X)^*$$

is canonically defined. In the next section we relate R_X to flat deformations of X over artin rings. For any artin local \mathbb{C} -algebra S , we set

$$J_{X,S} = J(\mathcal{T}_{X_\bullet} \otimes \mathfrak{m}_S)$$

and note that via the natural map $J_{X,S} \rightarrow J_X \otimes \mathfrak{m}_S$, any class $\epsilon \in \mathbb{H}^0(J_{X,S})$ yields a local homomorphism ('classifying map')

$${}^t\epsilon : R_X \rightarrow S.$$

As in the affine case, this construction may be extended to the case of maps. Thus let

$$f : X \rightarrow Y$$

be a morphism of schemes. Then we can choose respective affine coverings

$$X_\alpha \rightarrow P_\alpha, Y_\alpha \rightarrow Q_\alpha \text{ such that } f(X_\alpha) \subset Y_\alpha.$$

Then for each simplex ρ^k , the restriction of f yields a morphism

$$f_{\rho^k} : X_{\rho^k} \rightarrow Y_{\rho^k},$$

and for this we have an associated tangent dgla $\mathcal{T}_{f_{\rho^k}}$. Putting these together, we get a tangent SELA (with respect to the given affine coverings)

$$\mathcal{T}_f : \dots \rightarrow \mathcal{T}_{f_{\rho^k}} \rightarrow \dots$$

As before, \mathcal{T}_f is the mapping cone of

$$\mathcal{T}_{X\bullet} \oplus \mathcal{T}_{Y\bullet} \rightarrow f^! \mathcal{T}_{Y\bullet}.$$

Thus we have SELA morphisms

$$\mathcal{T}_f \rightarrow \mathcal{T}_Y, \mathcal{T}_f \rightarrow \mathcal{T}_X, f^! \mathcal{T}_Y[1] \rightarrow \mathcal{T}_f.$$

Correspondingly, we have a Jacobi-Bernoulli complex J_f , a deformation ring R_f together with maps $R_X \rightarrow R_f, R_Y \rightarrow R_f$.

Remark 2.2. When X is smooth, its tangent SELA is equivalent to a dgla, e.g. the Kodaira-Spencer algebra, a soft dgla resolution of the tangent sheaf.

3. DEFORMATIONS OF SCHEMES

3.1. Classification. Deformations of an algebraic scheme X/\mathbb{C} can be classified in terms of the associated tangent SELA \mathcal{T}_X and its Jacobi-Bernoulli cohomology. Consider first the case of an affine scheme $X \subset P$ (notations as in §2.1). Let S be a local artinian \mathbb{C} -algebra. Then a flat deformation of X over S is determined by, and determines, up to certain choices, an element

$$(3.32) \quad \phi \in \mathcal{T}_X^1(P) \otimes \mathfrak{m}_S = \mathcal{H}om^1(F_X^\bullet, F_{+X}^\bullet) \otimes \mathfrak{m}_S$$

known as a *Kodaira-Spencer cochain*, which satisfies the integrability condition

$$(3.33) \quad \partial\phi = -\frac{1}{2}[\phi, \phi].$$

The deformation corresponding to ϕ can be determined e.g. as the subscheme of $P \times \text{Spec}(S)$ having $(F_X^\bullet \otimes S, \partial + \phi)$ as resolution; we may denote this by X^ϕ .

Now globally, let X be an algebraic scheme over \mathbb{C} and as in §2.3 choose an affine embedding system

$$\iota_\rho : X_\rho \rightarrow P_\rho.$$

This gives rise as in §2.3 to a representative for the tangent SELA \mathcal{T}_X . Now suppose given a deformation of X over S as above. This restricts for each ρ to a deformation of X_ρ , whence a Kodaira-Spencer cochain $\phi_\rho \in \mathcal{T}_{X_\rho}^1(P_\rho) \subset \mathcal{T}_X^1(P_\bullet)$, satisfying an integrability condition as in (3.33), so that the restricted deformation of X_ρ is $X_\rho^{\phi_\rho}$. Moreover, the fact that ϕ_ρ and ϕ_σ restrict to equivalent deformations of $X_{\rho\sigma} \subset P_{\rho\sigma}$ yields an isomorphism

$$(3.34) \quad X_\rho^{\phi_\rho} \cap X_\sigma \simeq X_\sigma^{\phi_\sigma} \cap X_\rho;$$

both of these are closed subschemes of $P_{\rho\sigma} \times \text{Spec}(S)$ and the isomorphism (3.34) extends to an automorphism of $P_{\rho\sigma} \times \text{Spec}(S)$, necessarily of the form

$$\exp(t_{\rho\sigma}), t_{\rho\sigma} \in T_{P_{\rho\sigma}} \otimes \mathfrak{m}_S.$$

Then we get two resolutions of $X_{\rho}^{\phi_{\rho}} \cap X_{\sigma}$, the 'original' one with differential $\partial + \phi_{\rho}$, and the one pulled back from $X_{\sigma}^{\phi_{\sigma}} \cap X_{\rho}$, whose differential is $\partial + \phi_{\sigma}$. It is easy to see and well known that the two resolutions differ by an isomorphism of the form $\exp(u_{\rho\sigma})$ where $u_{\rho\sigma} \in \mathfrak{gl}^0(F_{X_{\rho}}) \otimes \mathfrak{m}_S$. Thus all in all there is a (uniquely determined) element

$$\psi_{\rho\sigma} \in \mathcal{T}_{X_{\rho\sigma}}^0 \otimes \mathfrak{m}_S = (T_{P_{\rho\sigma}} \oplus \mathfrak{gl}^0(F_{X_{\rho\sigma}})) \otimes \mathfrak{m}_S$$

such that

$$(3.35) \quad \exp(\psi_{\rho\sigma})(\partial + \phi_{\sigma}) \exp(-\psi_{\rho\sigma}) = \partial + \phi_{\rho}.$$

By construction, we clearly have

$$(3.36) \quad \exp(\psi_{\rho\sigma}) \exp(\psi_{\sigma\tau}) = \exp(\psi_{\rho\tau})$$

Thus, using (3.33, 3.35, 3.36), $\epsilon(\phi_{\bullet}, \psi_{\bullet})$ is a special multiplicative cocycle in the Jacobi-Bernoulli complex $J(\mathcal{T}_X) \otimes \mathfrak{m}_S$. Conversely, given a special multiplicative cocycle $\epsilon(\phi_{\bullet}, \psi_{\bullet})$ with values in S , the ϕ_{\bullet} data yields a collection of deformations of the affine pieces of X , while the ψ_{\bullet} glues these deformations together. These processes are inverse to each other up to an automorphism, and are precise mutual inverses when there are no automorphisms. Hence

Theorem 3.1. *Let X be an algebraic scheme over \mathbb{C} such that $H^0(\mathcal{T}_X) = 0$. Then for any local artin \mathbb{C} -algebra S , there is a bijection between the set of equivalence classes of flat deformations of X over S and the set of local homomorphisms from R_X to S .*

3.2. Obstructions. Let S be a local artin algebra and $I < S$ an ideal contained in the socle $\text{ann}_S(\mathfrak{m}_S)$ and $\bar{S} = S/I$. Let $\bar{\epsilon} = \epsilon(\bar{\phi}_{\bullet}, \bar{\psi}_{\bullet})$ be a special multiplicative cocycle with coefficients in \bar{S} . Let $\phi_{\bullet}, \psi_{\bullet}$ be arbitrary liftings of $\bar{\phi}_{\bullet}, \bar{\psi}_{\bullet}$ with coefficients in S . Thus, $\epsilon = \epsilon(\phi_{\bullet}, \psi_{\bullet})$ is not necessarily a cocycle. However, it is easy to check that the coboundary $\partial\epsilon$ lies in

$$(F_1 J_X)^1 \otimes I = (K^0(\mathcal{T}_X)^2 \oplus K^1(\mathcal{T}_X)^1 \oplus K^2(\mathcal{T}_X)^0) \otimes I$$

(because it dies mod I) and moreover, that $\partial\epsilon$ is a cocycle for $\text{tot}(K^{\cdot}(\mathcal{T}_X)^{\cdot}) \otimes I$ (because it is a cocycle for $J_{X,S}$). Thus, we obtain a cohomology class

$$(3.37) \quad \text{ob}(\bar{\phi}_{\bullet}, \bar{\psi}_{\bullet}) \in H^2(\mathcal{T}_X) \otimes I = H^2(\text{tot}(K^{\cdot}(\mathcal{T}_X)^{\cdot}) \otimes I).$$

This class is independent of choices and represents the obstruction to lifting $\bar{\epsilon}$ to a special multiplicative cocycle with coefficients in S .

3.3. Applications: relative obstructions, stable subschemes and surjections. Let $f : X \rightarrow Y$ be an embedding of a closed subscheme. We consider the question of 'relative obstructions' i.e. obstructions to lifting a given deformation of Y to a deformation of f . These are obstructions for the mapping cone of $\mathcal{T}_f \rightarrow \mathcal{T}_Y$, which is the same as that of $\mathcal{T}_X \rightarrow f^! \mathcal{T}_Y$. Locally, if $Y \rightarrow P$ is an affine embedding, such obstructions have values in $\text{Ext}^1(I_{X/Y}, \mathcal{O}_X)$, where $I_{X/Y} = I_{X/P}/I_{Y/P}$ is *independent of P* even as complex up to quasi-isomorphism (not just weak equivalence). Hence, global obstructions also have values in $\text{Ext}^1(I_{X/Y}, \mathcal{O}_X)$. In reasonably good cases though, the obstruction group can be narrowed considerably. The following result sharpens Thm. 1.1 of [5]

Theorem 3.2. *Let $X \subset Y$ be a closed subscheme having no component contained in the singular locus of Y . Then obstructions to deforming $X \rightarrow Y$ relative to deforming Y are in*

$$(3.38) \quad \text{Im}(\text{Ext}_Y^1(I_{X/Y}/I_{X/Y}^2, \mathcal{O}_X) \rightarrow \text{Ext}_Y^1(I_{X/Y}, \mathcal{O}_X)).$$

It follows easily, in particular, that if $X \rightarrow Y$ is moreover a regular embedding with normal bundle N and $H^1(X, N) = 0$, then $X \rightarrow Y$ is 'relatively unobstructed' or 'stable' relative to Y , i.e. deforms with every deformation of Y , and furthermore the Hilbert scheme of Y is smooth at the point corresponding to X . This generalizes a result of Kodaira [2] in the smooth case.

To sketch the proof, working locally, let F_Y be a free resolution of $I_{Y/P}$ and extend it to a free resolution F_X of $I_{X/P}$, such that, termwise,

$$F_X^i = F_Y^i \oplus F_{X/Y}^i$$

where $F_{X/Y}^i$ is a suitable free complement, $F_Y \rightarrow F_X$ is a map of complexes (though not a direct summand inclusion), and $F_{X/Y} \otimes \mathcal{O}_Y$, which is a quotient complex of F_X , is a free resolution of $I_{X/Y}$. We may also assume that $F_{X/Y}$ contains a subcomplex F_{X^2} (with termwise direct summands) resolving I_X^2 . A deformation of Y yields a linear map

$$v : F_Y^0 \rightarrow \mathcal{O}_X.$$

The obstruction to lifting this to a deformation of X is given by

$$v \circ \delta : F_{X/Y}^{-1} \rightarrow \mathcal{O}_X$$

where $\delta : F_{X/Y}^{-1} \rightarrow F_Y^0$ is the 'connecting map' from the resolution. δ takes a relation among generators of $I_X \bmod I_Y$ to the appropriate linear combination of generators of I_Y . Now, and this is the point, our assumption about singularities means that no generator of I_Y can be in I_X^2 , or more precisely, that

$$I_X^2 \cap I_Y \subset I_X I_Y.$$

This implies that $\delta(F_{X^2}^{-1}) \subset I_X F_Y^0$. Since v is \mathcal{O}_P -linear, it follows that the composition $v \circ \delta$ is zero on $F_{X^2}^{-1}$ and lives in the image as in (3.38). This

proves the result in the affine case, and the extension to the global case is straightforward. \square .

Next we consider an application to surjections is (compare [5], Thm 2.1):

Theorem 3.3. *Let $f : X \rightarrow Y$ be a projective morphism with*

$$f_*(\mathcal{O}_X) = \mathcal{O}_Y, R^1 f_*(\mathcal{O}_X) = 0.$$

Then f deforms with every deformation of X .

Proof. Here we use the fact that the mapping cone of $\mathcal{T}_f \rightarrow \mathcal{T}_X$ is equivalent to that of $\mathcal{T}_Y \rightarrow f^! \mathcal{T}_Y$. To prove the result it suffices to show the following (*) *the natural map $\mathcal{T}_Y \rightarrow f^! \mathcal{T}_Y$ induces a surjection on H^1 and an injection on H^2 .*

Indeed the H^1 -surjectivity property implies that any first-order deformation of X lifts to a deformation of f ; then the H^2 -injectivity property allows us to extend this inductively, via obstruction theory (§3.2), to n th order deformations. To prove (*), note that $f^! \mathcal{T}_Y$ can be represented by the tensor product $\mathcal{T}_Y \otimes \mathrm{XC}(\mathcal{O}_X)$, which is a double complex with terms $\mathcal{T}_Y^i \otimes \mathrm{XC}^j(\mathcal{O}_X)$. Then the spectral sequence of a double complex (or an elementary substitute) yields our conclusion. \square

Morphisms f satisfying the hypotheses of the Theorem occur in diverse situations, e.g. regular fibre spaces and resolutions of rational singularities. The Theorem says that those schemes X which admit a structure such as f form an open subset of the moduli of X .

Theorem 3.4. *Let $f : X \rightarrow Y$ be a proper surjective morphism étale in codimension 1 where X is normal. Then any deformation of f is determined by the associated deformation of Y .*

Proof. It will suffice to prove that the mapping cone $\mathcal{T}_X \rightarrow f^! \mathcal{T}_Y$ is exact in degrees ≤ 1 . Working locally, we may view X as a subscheme of $Y \times R$, R an affine space, and consider a free resolution J^\bullet of the ideal of X in $Y \times R$. Let K be the kernel of the natural map $\mathcal{H}om(J^0, \mathcal{O}_X) \rightarrow \mathcal{H}om(J^{-1}, \mathcal{O}_X)$. Then K is a torsion-free \mathcal{O}_X module and the natural map $k : T_R \otimes \mathcal{O}_X \rightarrow K$ is an isomorphism in codimension 1 by our assumption that f is étale in codimension 1. As X is normal, this easily implies k is an isomorphism, which proves our assertion. \square

It follows, e.g. that a small resolution of a singularity Y is locally uniquely determined by Y , i.e. cannot be deformed without deforming Y . Smallness is of course essential here. Under stronger hypotheses on the size of the exceptional locus, we can actually identify the deformations of f and Y (compare [5], Thm. 3.5):

Theorem 3.5. *Let $f : X \rightarrow Y$ be a proper surjective morphism étale in codimension 2 with X smooth. Then any deformation of Y lifts to a deformation of f .*

Proof. The idea is to relate deformations to Kähler differentials, e.g. relate $H^i(\mathcal{T}_Y)$ to $\text{Ext}^i(\Omega_Y, \mathcal{O}_Y)$, so that we may apply Ischebeck's Lemma ([1],[4] p. 104). We work in the affine setting, with $Y \rightarrow P$ an affine embedding. Let F^\cdot be a free resolution of $I = I_{Y/P}$ and consider the complex

$$(3.39) \quad \hat{\Omega}_Y : \dots \rightarrow F^{-1} \otimes \mathcal{O}_Y \rightarrow F^0 \otimes \mathcal{O}_Y \rightarrow \Omega_P \otimes \mathcal{O}_Y$$

where the last map corresponds to the derivative map $I \rightarrow \Omega_P \otimes \mathcal{O}_Y$. By definition, applying $\mathcal{H}om(\cdot, \mathcal{O}_Y)$ to this complex yields the tangent SELA \mathcal{T}_Y . Then $\hat{\Omega}_Y$ induces a complex of free \mathcal{O}_X -modules

$$(3.40) \quad f^! \hat{\Omega}_Y : \dots \rightarrow F^{-1} \otimes \mathcal{O}_X \rightarrow F^0 \otimes \mathcal{O}_X \rightarrow \Omega_P \otimes \mathcal{O}_X$$

and applying $\mathcal{H}om(\cdot, \mathcal{O}_X)$ to this yields $f^! \mathcal{T}_Y$. Now let J^\cdot be an \mathcal{O}_Y -free resolution of I/I^2 , and consider the complex

$$(3.41) \quad \tilde{\Omega}_Y : \dots \rightarrow J^{-1} \rightarrow J^0 \rightarrow \Omega_P \otimes \mathcal{O}_Y$$

We may assume that

$$(3.42) \quad J^i \simeq F^i \otimes \mathcal{O}_Y, i = 0, 1; \quad F^{-2} \otimes \mathcal{O}_Y \hookrightarrow J^{-2}.$$

We have a natural map $\hat{\Omega}_Y \rightarrow \tilde{\Omega}_Y$ hence $f^! \hat{\Omega}_Y \rightarrow f^! \tilde{\Omega}_Y$, and these induce maps

$$(3.43) \quad \text{Ext}^i(\tilde{\Omega}_Y, \mathcal{O}_Y) \rightarrow H^i(\mathcal{T}_Y), \text{Ext}^i(f^! \tilde{\Omega}_Y, \mathcal{O}_X) \rightarrow H^i(f^! \mathcal{T}_Y)$$

that are bijective for $i \leq 1$, injective for $i = 2$. Note that $H^0(f^! \tilde{\Omega}_Y) = f^* \Omega_Y$, while for $i < 0$, $H^i(f^! \tilde{\Omega}_Y)$ is supported on the exceptional locus of f . By Ischebeck, it follows that

$$(3.44) \quad \text{Ext}^i(f^! \tilde{\Omega}_Y, \mathcal{O}_X) = \text{Ext}^i(f^* \Omega_Y, \mathcal{O}_X), i \leq 2.$$

By Ischebeck again, the natural map

$$(3.45) \quad \text{Ext}^i(\Omega_X, \mathcal{O}_X) = H^i(\mathcal{T}_X) \rightarrow \text{Ext}^i(f^* \Omega_Y, \mathcal{O}_X)$$

is bijective for $i \leq 1$ and injective for $i = 2$, because the cokernel of $f^* \Omega_Y \rightarrow \Omega_X$ is supported on the exceptional locus of f .

Now given (3.43-3.45), the completion of the proof is boilerplate. Thus suppose given a 1st order deformation α of Y . By (3.43) $f^! \alpha$ yields an element of $\text{Ext}^1(f^! \tilde{\Omega}_Y, \mathcal{O}_X)$ hence by (3.44) an element of $\text{Ext}^1(f^* \Omega_Y, \mathcal{O}_X)$, which by (3.45) comes from an element β of $H^1(\mathcal{T}_X)$, i.e. a 1st order deformation of X compatible with α . Thus, α lifts to a deformation of f . If α lifts to 2nd order, a suitable obstruction element vanishes, and applying a similar argument to the obstruction of β then shows that this obstruction must vanish too. Continuing in this manner, we show that any deformation of Y , of any order, lifts to a deformation of f . \square

The Theorem does not extend to morphisms with a codimension-2 exceptional locus, such as a small resolution of a 3-fold ODP.

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